

Robot Javelin Puzzle – Final Proof

1 Nash Equilibrium of the Original Game

Consider the original game without cheating. Each player adopts a threshold strategy: keep the first throw if it is at least t , otherwise reroll.

Let Y denote the opponent's final score and define the CDF

$$F_Y(y) = \mathbb{P}(Y \leq y).$$

Distribution of Y

If $0 \leq y \leq t$, the opponent must have rerolled, hence

$$F_Y(y) = \mathbb{P}(Y_2 \leq y) \mathbb{P}(Y_1 \leq t) = ty.$$

If $t \leq y \leq 1$, either the first throw is kept or rerolled:

$$F_Y(y) = y(1+t) - t.$$

Thus,

$$F_Y(y) = \begin{cases} ty, & 0 \leq y \leq t, \\ y(1+t) - t, & t \leq y \leq 1. \end{cases}$$

Expected Value

Using the tail-integral formula,

$$\mathbb{E}[Y] = \int_0^1 (1 - F_Y(y)) dy = \frac{1}{2} + \frac{t}{2} - \frac{t^2}{2}.$$

Indifference Condition

At equilibrium, keeping or rerolling at $x = t$ must give the same win probability:

$$t^2 = 1 - \mathbb{E}[Y].$$

This yields

$$t^2 + t - 1 = 0, \quad t = \frac{\sqrt{5} - 1}{2} \approx 0.618.$$

2 Cheating Player's Optimal Strategy

The cheating player (Spears) observes whether $J_1 \geq c$ with

$$c = \frac{\sqrt{5} - 1}{2}.$$

Case 1: $J_1 < c$

Java must reroll, so her final score is uniform on $[0, 1]$. Spears compares:

- Keep s : win probability = s ,
- Reroll: win probability = $\frac{1}{2}$.

Thus,

$$t_0 = \frac{1}{2}.$$

Case 2: $J_1 \geq c$

Java keeps her first throw, hence $J \sim U[c, 1]$. The CDF is

$$F(x) = \begin{cases} 0, & x < c, \\ \frac{x - c}{1 - c}, & x \geq c. \end{cases}$$

The reroll win probability is

$$\int_0^1 F(u) du = \frac{1 - c}{2}.$$

Indifference gives

$$\frac{s - c}{1 - c} = \frac{1 - c}{2} \Rightarrow t_1 = c + \frac{(1 - c)^2}{2} \approx 0.690983.$$

3 Java-lin's Optimal Counter-Strategy

Focus on the branch $J_1 < c$, where Spears uses $t_0 = \frac{1}{2}$.

Spears' Final Distribution

$$H_{0.5}(x) = \begin{cases} 0.5x, & x < 0.5, \\ 1.5x - 0.5, & x \geq 0.5. \end{cases}$$

Reroll Win Probability

$$\mathbb{P}(\text{reroll wins}) = \int_0^1 H_{0.5}(u) du = \frac{3}{8}.$$

Indifference Condition

Let a_0 be Java's threshold in this branch. Indifference requires

$$H_{0.5}(a_0) = \frac{3}{8}.$$

Since $a_0 > 0.5$,

$$1.5a_0 - 0.5 = \frac{3}{8} \Rightarrow a_0 = \frac{7}{12}.$$

4 Final Winning Probability

Partition $J_1 \in [0, 1]$:

A. $J_1 \in [0, \frac{7}{12})$

Java rerolls:

$$I_1 = \frac{3}{8} \cdot \frac{7}{12}.$$

B. $J_1 \in [\frac{7}{12}, c)$

Java keeps:

$$I_2 = \int_{7/12}^c \left(\frac{3}{2}j - \frac{1}{2} \right) dj.$$

C. $J_1 \in [c, 1]$

Spears uses t_1 :

$$I_3 = \int_c^{t_1} t_1 j \, dj + \int_{t_1}^1 ((1 + t_1)j - t_1) \, dj.$$

Final Result

$$\mathbb{P}(\text{Java wins}) = I_1 + I_2 + I_3 \approx 0.4939370904.$$

Conclusion

$\mathbb{P}(\text{Java-lin wins}) \approx 49.3937\%$
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