Mo Tu We Th Fr Sa Su

Memo No. 1
Date 25 / 12 / 06

? why is the Nash Equilibrium of the orignal game equal to 0.618 ?

Proof. Suppose that the NE $= t$, and consider player Y., $Y_1 \sim U[0,1]$, and if $Y_1 \leq t$, then $Y_2 \sim U[0,1]$.

Let $F_t(y) = P(Y \leq y)$.

Situation 1: $0 \leq y \leq t$

It can't be the case that $Y_1 < t$ and the player kept it., thus it must be the case that the player re-rolled, and $y_2 < t$. Therefore:

$$F_t(y) = \underbrace{P(Y_1 < t)}_{t}\underbrace{P(Y_2 \leq y)}_{y} = t \cdot y$$

Situation 2: $t \leq y \leq 1$

It could be either y_1 or y_2. To be more explicit, either y_1 then stop, or $y_1 \leq t$ then $y_2 > t$.

Memo No. 2
Date / /

$P(y_1 \geq t) = y - t$

$P(y_1 \leq t) = t$

$P(y_2 > t \mid y_1 \leq t) = t.$

$F_t(y) = (y-t) + t \cdot y$

$= y(1+t) - t.$

Summary of situation 1 and 2:

$$F_t(y) = \begin{cases} ty & 0 \leq y \leq t \\ y(1+t)-t & t \leq y \leq 1 \end{cases}$$

Now consider player X.

If player X rerolls, then $X' \sim U[0,1]$

$$P(\text{player X rerolls and wins}) = P(X' > Y)$$

⚠ The law of total expectation says that:

$$\mathbb{E}[X] = \mathbb{E}[P(X \mid Y)].$$

So:

$P(X' \leq Y \mid Y) = Y$

cuz $X' \sim U[0,1]$

$$P(X' > Y) = \mathbb{E}[P(X' > Y \mid Y)]$$

$$= \mathbb{E}[1 - P(X' \leq Y \mid Y)]$$

$$= \mathbb{E}[1 - Y]$$

continue: $P(X'>Y) = \mathbb{E}[1-Y] = 1-\mathbb{E}[Y]$

Given the Tail-Integral Formula.

$$\mathbb{E}[Y] = \int_0^{\infty} P(Y>y)\, dy.$$

So: $$\mathbb{E}[Y] = \int_0^1 (1-F_t(y))\, dy$$

$$\mathbb{E}[Y] = \int_0^t 1-F_t(y)\, dy + \int_t^1 1-F_t(y)\, dy.$$

Recall that: $$F_t(y) = \begin{cases} ty & \text{for } 0 \le y < t \\ y(1+t)-t & \text{for } t \le y \le 1 \end{cases}$$

the 1st integral becomes (I_1):

$$I_1 = \int_0^t 1-F_t(y)\, dy = \int_0^t 1-ty\, dy = \int_0^t 1\, dy - \int_0^t ty\, dy$$

$$\boxed{I_1 = t - \frac{t^3}{2}}.$$

The 2nd integral, I_2, becomes:

$$I_2 = \int_t^1 1-[y(1+t)-t]\, dy.$$

$$= \int_t^1 (1+t)(1-y)\, dy$$

$$= (1+t)\int_t^1 1-y\, dy$$

$$= (1+t)\left(\frac{1}{2} - t + \frac{t^2}{2}\right)$$

$$= \cancel{\ldots}$$

$$= \frac{1}{2} - \frac{t}{2} - \frac{t^2}{2} + \frac{t^3}{2}.$$

Combine I_1 and I_2:

$$\mathbb{E}[Y] = I_1 + I_2 = \left(t - \frac{t^3}{2}\right) + \left(\frac{1}{2} - \frac{t}{2} - \frac{t^2}{2} + \frac{t^3}{2}\right)$$

$$\boxed{\mathbb{E}[Y] = \frac{1}{2} + \frac{t}{2} - \frac{t^2}{2}}$$

Therefore: $P(X' > Y) = 1 - E[Y]$

$$= \frac{t^2 - t + 1}{2}$$

Than, $P(X$ wins given no X reroll$) = P(Y < x)$

$$= F_t(x)$$

From previous, we know that:

$$F_t(x) = \begin{cases} tx & 0 \leq x \leq t \\ x(1+t) - t & t \leq x \leq 1 \end{cases}$$

We need to find a t such that when $x = t$, we are indifferent between keep and reroll.

Note that $F_t(x)$ becomes t^2 when $x = t$.

So:
$$t^2 = \frac{t^2 - t + 1}{2}$$

$$t = \frac{\sqrt{5} - 1}{2} \approx 0.618.$$

③ What is the optimal strategy if one of the players could cheat, i.e. Stage II.

Proof. The optimal strategy is to set the cheating observe window to be 0.618. Intuition, if you set the threshold to be 0.618, then you know the other player's throws must come from either i) $U[0.618, 1]$ if her first throw is above 0.618 or ii) $U[0,1]$ if her first throw is below 0.618.

- What is the cheating player's optimal strategy in this case?
- If the cheating player's first throw is < 0.691 and the opponent is higher than 0.618, then keep, otherwise reroll. If the opponent's first throw is less than 0.618, then the cheater would keep anything above 0.5.

why?. say opponent is Y, then: , oh and the optimal threshold is a.

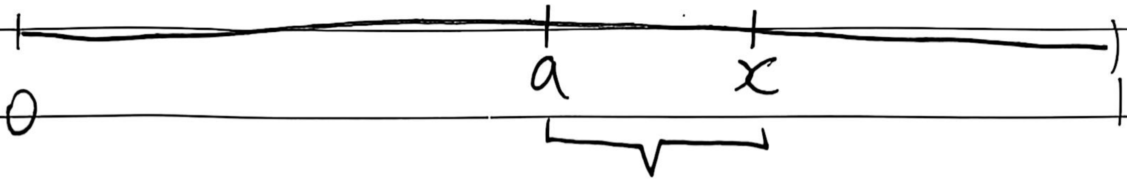
$\mathbb{E}[Y \mid Y\text{'s first throw is above } a \text{ and } a \text{ above } 0.618] = \frac{a+1}{2}$
we recall this situation I, i.e. $Y_1 > a$.

Memo No. 2
Date / /

$P_1 \doteq$ X wins, the $P_1 = \frac{a^2}{2} \doteq 0.191$,
if X also above a.

Basically the half way between 0.618 and 1.
$(1-0.618) \div 2 = 0.191$.

$$F_1(x) = P(Y \leq x \mid I=1) = \begin{cases} 0 & x < a \\ \frac{x-a}{1-a} & a \leq x \leq 1 \end{cases}$$



Y needs to fall here for X to win

$$\frac{x-a}{1-a} = \frac{a^2}{2} \Rightarrow x = a + \frac{(1-a)^2}{2} = \cancel{0.618}$$

when $a = 0.618$, $x = 0.6909$.

Hence, the threshold is 0.6909 ish.

Mo Tu We Th Fr Sa Su

Memo No. 1
Date / /

(?) what is the optimal strategy if one player cheats and the other player knows the former cheats without the former knows the latter cheats?

so if Javalin rolled anything greater than 0.618 she is going to keep. So let's consider the case where she rolled less than 0.618.

In this case, the cheater, spears, is going to decide based on a threshold of 0.5.

Recall that:

$$F_t(y) = \begin{cases} ty & 0 \le y \le t \\ y(1+t)-t & t \le y \le 1 \end{cases}$$

when $t = 0.5$:

$$F_{0.5}(y) = \begin{cases} 0.5y & 0 \le y \le 0.5 \\ y(1+0.5)-0.5 & 0.5 \le y \le 1 \end{cases}$$

Memo No. 2
Date / /

In this case, Javalin's win rate is $\int_0^1 H_{0.5}(u)\,du$ reroll.

$$= \int_0^{0.5} 0.5u\,du + \int_{0.5}^{1} (1.5u - 0.5)\,du = \boxed{\frac{3}{8}}$$

Prob Javalin wins when keep, using threshold α is $F_{0.5}(\alpha)$.

keep should equal to reroll. → assuming $\alpha > 0.5$

$$F_{0.5}(\alpha) = \frac{3}{8}$$

$$1.5\alpha - 0.5 = \frac{3}{8}$$

$$\boxed{\alpha = \frac{7}{12}}$$

Here Javalin should keep anything above $\frac{7}{12}$

(7) What is Javalin's final win rate?

A: $j \in [0, a_0 = \frac{7}{12}]$

In this case, Spears uses $t_0 = \frac{1}{2}$, and Javalin rerolls.

$$\int_0^1 H_{0.5}(u)\,du = \frac{3}{8}$$

$$I_1 = \int_0^{a_0} \frac{3}{8}\,dj = \frac{3}{8} a_0 \quad \frac{7}{12}$$

B: $j \in [a_0, c]$, $a_0 = \frac{7}{12}$, $c = 0.618$

In this case, Spears uses $t_0 = \frac{1}{2}$, and Javalin keeps.

$$H_{0.5}(j) = \frac{3}{2}j - \frac{1}{2},$$

$$I_2 = \int_{a_0}^{c} \left(\frac{3}{2}j - \frac{1}{2}\right) dj$$

Memo No. 2
Date / /

C: $j \in [c, 1]$

spers uses $t_0 = 0.690983$, Java keeps.

if $j \in [c, t_1)$, $H_{t_1}(j) = t_1 j$

$j \in [t_1, 1]$, $H_{t_1}(j) = (1+t_1)j - t_1$

$$I_3 = \int_c^{t_1} t_1 j \, dj + \int_{t_1}^{1} ((1+t_1)j - t_1) \, dj$$

$P(\text{Java wins}) = I_1 + I_2 + I_3 = \cancel{49.3937}$

$\boxed{49.39370903646491\%}$